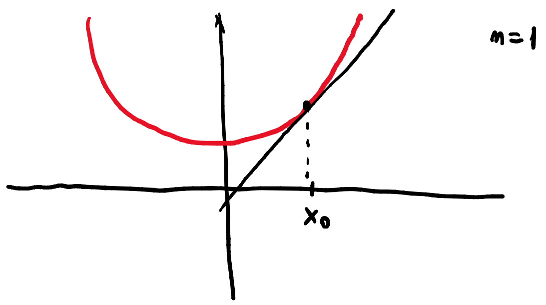
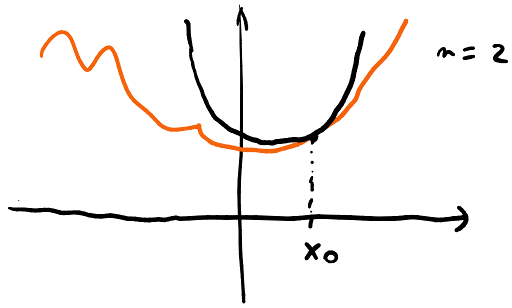
Limiti con polinomi di Taylor.

Il polinomio di Taylor di ordine $n \in \mathbb{N}$ e centro x_0 di una funzione f è il polinomio di grado $\leq n$ che approssima f vicino al punto x_0 meglio di qualsiasi altro polinomio di grado $\leq n$.





Espressione dei polinomi di Taylor:

$$P_{n,f,x_0}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k \quad \text{dove} \quad k! \begin{cases} 1 & \text{se } k=0 \\ 1\cdot 2\cdot \ldots \cdot k & \text{se } k\geq 1. \end{cases}$$

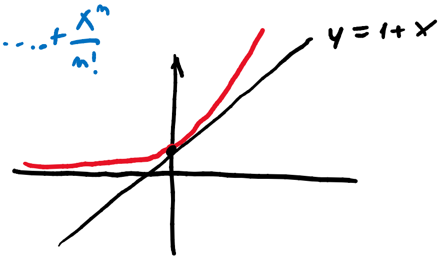
$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \ldots$$

$$+ \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

ESEMPI

$f(x) = e^x \quad , \quad x_0 = 0$

$$P_{n,f,0}(x) = \sum_{k=0}^{n} \frac{x^k}{k!} = 1 + x + \frac{1}{2}x^2 + \frac{x^3}{6} + \ldots + \frac{x^n}{n!}$$



$$P_{1,f,0}(x) = \sum_{k=0}^{1} \frac{x^k}{k!} = 1 + x$$

$$P_{2,f,0}(x) = 1 + x + \frac{x^2}{2!} = 1 + x + \frac{x^2}{2}$$

$$P_{3,f,0}(x) = 1 + x + \frac{1}{2}x^2 + \frac{x^3}{6}$$

$$P_{4,f,0}(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{x^4}{24}$$

2) $f(x) = \log(1+x)$

$$P_{n,f,0}(x) = \sum_{n=1}^{n} (-1)^{k+1} \frac{x^k}{k}.$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots + (-1)^{n+1} \frac{x^n}{n}.$$

$$P_{1,f,0}(x) = x$$

$$P_{2,f,0}(x) = x - \frac{x^2}{2}$$

$$P_{3,f,0}(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$$

3) $f(x) = \frac{1}{1-x}$

$$P_{n,f,0}(x) = \sum_{k=0}^{n} x^k = 1 + x + x^2 + x^3 + \dots + x^n.$$

$$P_{1,f,0}(x) = 1 + x$$

$$P_{2,f,0}(x) = 1 + x + x^2$$

$$P_{3,f,0}(x) = 1 + x + x^2 + x^3$$

4) $f(x) = \sin x$

$$P_{2n+1,f,0}(x) = \sum_{k=0}^{n} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$P_{1,f,0}(x) = x$$

$$P_{2,f,0}(x) = x$$

$$P_{3,f,0}(x) = x - \frac{x^3}{3!} = x - \frac{x^3}{6}$$

$$P_{4,f,0}(x) = x - \frac{x^3}{6}$$

$$P_{5,f,0}(x) = x - \frac{x^3}{6} + \frac{x^5}{120}$$

5) $f(x) = \cos x$

$$P_{2m,0,f}(x) = \sum_{k=0}^{m} (-1)^k \frac{x^{2k}}{(2k)!}$$

$$= 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots + (-1)^m \frac{x^{2m}}{(2m)!}$$

$P_{1,0,f}(x) = 1$

$P_{2,0,f}(x) = 1 - \frac{x^2}{2}$

$P_{3,0,f}(x) = 1 - \frac{x^2}{2}$

$P_{4,0,f}(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$

6) $f(x) = \arctan x$

$$P_{2m+1}(x) = \sum_{k=0}^{m} (-1)^k \frac{x^{2k+1}}{2k+1}$$

$$= x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots + (-1)^m \frac{x^{2m+1}}{2m+1}$$

7) $(1+x)^\alpha$

$$P_{m,f,0}(x) = \sum_{k=0}^{m} x^k \binom{\alpha}{k} \qquad \binom{\alpha}{k} = \begin{cases} 1 & \text{se } k = 0 \\ \dfrac{\alpha(\alpha-1)\cdot\dots\cdot(\alpha-k+1)}{k!} & \text{se } k \geq 1 \end{cases}$$

$$P_{m,f,0}(x) = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2} x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{6} x^3 + \dots$$

Caso particolare

$f(x) = \sqrt{1+x} = (1+x)^{\frac{1}{2}} \qquad \alpha = \frac{1}{2}$

$$P_{m,f,0}(x) = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2} x^2 + \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{6} x^3 + \dots$$

$$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 + \dots,$$

Formula di Taylor con resto di Peano

$$f(x) = P_{n,f,0}(x) + \underbrace{o(x^n)}$$ per $x \to 0$.

quantità trascurabile rispetto a x^n

cioè $\lim_{x \to 0} \frac{o(x^n)}{x^n} = 0$

(simbolo di Landau)

Proprietà dei simboli di Landau

$o(x^n) + o(x^n) = o(x^n)$

$o(x^n) - o(x^n) = o(x^n)$

$o(2x^n) = o(x^n)$

$2\,o(x^n) = o(x^n)$

$o(x^n)\,o(x^m) = o(x^{n+m})$

$o(x^n) + o(x^m) = o(x^{\min\{m,n\}}$

ESEMPI DI CALCOLO DI LIMITI

1) $\lim_{x \to 0} \frac{\sin x - \arctan x}{x^3}$ f.i. $\frac{0}{0}$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$\arctan x = x - \frac{x^3}{3} + o(x^3)$$

Numeratore:

$\sin x - \arctan x$

$= x - \frac{x^3}{6} + o(x^3) - \left(x - \frac{x^3}{3} + o(x^3)\right)$

$= \cancel{x} - \frac{x^3}{6} + o(x^3) - \cancel{x} + \frac{x^3}{3} - o(x^3)$

$= \left(\frac{1}{3} - \frac{1}{6}\right) x^3 + o(x^3) = \frac{1}{6} x^3 + o(x^3)$

$$\lim_{x\to 0} \frac{\operatorname{sen} x - \arctan x}{x^3} = \lim_{x\to 0} \frac{\frac{1}{6}x^3 + o(x^3)}{x^3} = \lim_{x\to 0} \frac{1}{6} + \frac{o(x^3)}{x^3}$$

$$= \frac{1}{6}$$

E, se avessimo preso i polinomi di grado 1 invece che 3?

$$\sin x = x + o(x)$$

$$\arctan x = x + o(x)$$

$$\sin x - \arctan x = o(x)$$

Non ci permette di trovare la parte principale del numeratore

$$\sin x = x - \frac{1}{6}x^3 + o(x^3)$$

$$\arctan x = x + o(x)$$

$$\sin x - \arctan x = -\frac{1}{6}x^3 + o(x^3) - o(x)$$

$$= -\frac{1}{6}x^3 + \underbrace{o(x)}$$

Non è trascurabile rispetto a x^3.

Anche in questo caso, non possiamo concludere perché non abbiamo trovato la parte principale del numeratore.

Prendendo i polinomi di Taylor di ordine 5 invece possiamo concludere:

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$$

$$\arctan x = x - \frac{1}{3}x^3 + \frac{x^5}{5} + o(x^5)$$

$$\sin x - \arctan x = \frac{1}{6}x^3 + \underbrace{\left(\frac{1}{120} - \frac{1}{5}\right)x^5 + o(x^5)}_{o(x^3)}$$

$$= \frac{1}{6}x^3 + o(x^3).$$

ESERCIZIO

$$\lim_{x\to 0} \frac{e^{-\frac{x^2}{2}} - \cos x}{x^2 - \log(1+x^2)} \qquad \frac{1-1}{0} = \frac{0}{0} \quad f.i.$$

Denominatore:

$$\log(1+x^2) \overset{y=x^2}{=} \log(1+y)$$
$$= y - \frac{1}{2}y^2 + \frac{1}{3}y^3 + o(y^3)$$
$$= x^2 - \frac{1}{2}(x^2)^2 + \frac{1}{3}(x^2)^3 + o((x^2)^3)$$
$$= x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 + o(x^6)$$

$$x^2 - \log(1+x^2) = x^2 - \left(x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 + o(x^6)\right)$$
$$= \cancel{x^2} - \cancel{x^2} + \frac{1}{2}x^4 - \frac{1}{3}x^6 + o(x^6)$$
$$= \frac{1}{2}x^4 - \frac{1}{3}x^6 + o(x^6)$$
$$= \frac{1}{2}x^4 + o(x^4)$$

Numeratore: $e^{-\frac{x^2}{2}} - \cos x$

$$e^{-\frac{x^2}{2}} \overset{y=-\frac{x^2}{2}}{=} e^y = 1 + y + \frac{1}{2}y^2 + o(y^2)$$
$$= 1 - \frac{x^2}{2} + \frac{1}{2}\left(-\frac{x^2}{2}\right)^2 + o\left(\left(-\frac{x^2}{2}\right)^2\right)$$
$$= 1 - \frac{x^2}{2} + \frac{x^4}{8} + o(x^4)$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{x^4}{24} + o(x^4)$$

$$e^{-\frac{x^2}{2}} - \cos x = \frac{x^4}{8} - \frac{x^4}{24} + o(x^4)$$
$$= +\frac{1}{12}x^4 + o(x^4)$$

Quindi
$$\lim_{x\to 0} \frac{\frac{1}{12}x^4 + o(x^4)}{\frac{1}{2}x^4 + o(x^4)} = \lim_{x\to 0} \frac{\frac{1}{12}\cancel{x^4}}{\frac{1}{2}\cancel{x^4}} =$$
$$= \frac{1}{12}\cdot 2 = \frac{1}{6}$$

- $$\lim_{x\to 0} \frac{e^{-\frac{x^2}{2}} - \cos x}{(x^2 - \log(1+x^2))^2}$$ f.i. $\frac{1-1}{0^2} = \frac{0}{0}$

Denominatore:

$$\left(x^2 - \log(1+x^2)\right)^2 = \left(\frac{1}{2}x^4 + o(x^4)\right)^2$$
$$= \frac{1}{4}x^8 + x^4 o(x^4) + o(x^8)$$
$$= \frac{1}{4}x^8 + o(x^8) + o(x^8)$$
$$= \frac{1}{4}x^8 + o(x^8)$$

Quindi:

$$\lim_{x\to 0} \frac{\frac{1}{12}x^4 + o(x^4)}{\frac{1}{4}x^8 + o(x^8)} = \lim_{x\to 0} \frac{\frac{1}{12}x^4}{\frac{1}{4}x^8} = \lim_{x\to 0} \frac{1}{3}\frac{1}{x^4} = +\infty.$$

ESEMPIO

$$\lim_{x\to 0} \frac{\sin^2 x - x\arctan x}{e^{2x^2}\cos x - 1 - \frac{3}{2}x^2}$$ f.i.: $\frac{0}{0}$

Numeratore:

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$
$$\sin^2 x = \left(x - \frac{x^3}{6} + o(x^3)\right)^2$$
$$= x^2 - \frac{x^4}{3} + o(x^4) + \left(\frac{x^3}{6}\right)^2 + o(x^6) + \underbrace{(o(x^3))^2}_{=o(x^6)}$$
$$= x^2 - \frac{x^4}{3} + o(x^4)$$

$$\arctan x = x - \frac{x^3}{3} + o(x^3)$$
$$x\arctan x = x^2 - \frac{x^4}{3} + o(x^4)$$

Devo prendere più termini:

$$\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} + o(x^5)$$

$$\sin^2 x = x^2 - \frac{x^4}{3} + \frac{x^6}{60} + o(x^6) + \frac{x^6}{36}$$

$$= x^2 - \frac{x^4}{3} + \frac{2}{45}x^6 + o(x^6)$$

$$x \arctan x = x\left(x - \frac{x^3}{3} + \frac{x^5}{5} + o(x^5)\right)$$

$$= x^2 - \frac{x^4}{3} + \frac{x^6}{5} + o(x^6)$$

$$\sin^2 x - x\arctan x = \left(\frac{2}{45} - \frac{1}{5}\right)x^6 + o(x^6)$$

$$= -\frac{7}{45}x^6 + o(x^6)$$

<u>Denominatore</u>: $e^{2x^2}\cos x - 1 - \frac{3}{2}x^2$.

$$e^{2x^2} \overset{y=2x^2}{=} 1 + y + \frac{1}{2}y^2 + o(y^2)$$

$$= 1 + 2x^2 + \frac{1}{2}(2x^2)^2 + o((2x^2)^2)$$

$$= 1 + 2x^2 + 2x^4 + o(x^4)$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$$

$$e^{2x^2}\cos x = (\underline{1} + \underline{2x^2} + \underline{2x^4} + o(x^4))\left(\underline{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)}\right)$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4) + 2x^2 - x^4 + 2x^4$$

$$= 1 + \left(2 - \frac{1}{2}\right)x^2 + \left(\frac{1}{24} + 1\right)x^4 + o(x^4)$$

$$= 1 + \frac{3}{2}x^2 + \frac{25}{24}x^4 + o(x^4)$$

$$e^{2x^2}\cos x - 1 - \frac{3}{2}x^2 = \frac{25}{24}x^4 + o(x^4)$$

Conclusione:

$$\lim_{x\to 0} \frac{-\frac{7}{45}\cdot x^6 + o(x^6)}{\frac{25}{24}x^4 + o(x^4)} = \lim_{x\to 0} \frac{-\frac{7}{45}}{\frac{25}{24}}x^2 = 0.$$

ESERCIZI

1) $$\lim_{x\to 0} \frac{(e^{2x}-1)^2 - 4x\sin x}{x^3+x^4}$$

Numeratore:

$e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$

$e^{2x} = 1 + 2x + \frac{1}{2}(2x)^2 + o((2x)^2) = 1 + 2x + 2x^2 + o(x^2)$

$(e^{2x}-1)^2 = (2x + 2x^2 + o(x^2))^2 = 4x^2 + 8x^3 + o(x^3)$

$4x\sin x = 4x\left(x - \frac{x^3}{6} + o(x^3)\right) = 4x^2 - \frac{2}{3}x^4 + o(x^4) = 4x^2 + o(x^3)$

$(e^{2x}-1)^2 - 4x\sin x = 4x^2 + 8x^3 + o(x^3) - (4x^2 + o(x^3))$

$= 8x^3 + o(x^3)$

Denominatore: $x^3 + x^4 = x^3 + o(x^3)$

$$\lim_{x\to 0} \frac{(e^{2x}-1)^2 - 4x\sin x}{x^3+x^4} = \lim_{x\to 0} \frac{8x^3+o(x^3)}{x^3+o(x^3)} = \lim_{x\to 0} \frac{8\cancel{x^3}}{\cancel{x^3}} = 8.$$

2) $$\lim_{x\to 0} \frac{x\cos(2x) + e^{-x} - 1}{(\log(1+\sqrt{x}) - \sqrt{x})^2}$$

Numeratore:

$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$

$\cos 2x = 1 - \frac{1}{2}(2x)^2 + \frac{1}{24}(2x)^4 + o((2x)^4) = 1 - 2x^2 + \frac{2}{3}x^4 + o(x^4)$

$e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$

$e^{-x} = 1 - x + \frac{1}{2}(-x)^2 + o((-x)^2) = 1 - x + \frac{1}{2}x^2 + o(x^2)$

$x\cos(2x) + e^{-x} - 1 = x(1 - 2x^2 + o(x^2)) + 1 - x + \frac{1}{2}x^2 + o(x^2) - 1$

$= \cancel{x} - 2x^3 + o(x^3) \cancel{+1} - \cancel{x} + \frac{1}{2}x^2 + o(x^2) \cancel{-1}$

$= \frac{1}{2}x^2 - 2x^3 + o(x^3) + o(x^2)$

$= \frac{1}{2}x^2 + o(x^2)$

$\log(1+x) = x - \frac{1}{2}x^2 + o(x^2)$

$$\log(1+\sqrt{x}) = \sqrt{x} - \frac{1}{2}x + o(x)$$

$$\left(\log(1+\sqrt{x}) - \sqrt{x}\right)^2 = \left(-\frac{1}{2}x + o(x)\right)^2 = \frac{1}{4}x^2 + o(x^2) + o(x^2)$$
$$= \frac{1}{4}x^2 + o(x^2)$$

$$\lim_{x\to 0} \frac{x\cos(2x) + e^{-x} - 1}{(\log(1+\sqrt{x}) - \sqrt{x})^2} = \lim_{x\to 0} \frac{\frac{1}{2}x^2 + o(x^2)}{\frac{1}{4}x^2 + o(x^2)} = \frac{\frac{1}{2}}{\frac{1}{4}} = 2$$

3) $$\lim_{x\to 0} \frac{e^{\frac{x}{2}} + \cos(\sqrt{x}) - 2}{\sqrt[3]{1+x^2} - e^{x^2}}$$

$$e^x = 1 + x + \frac{1}{2}x^2 + o(x^2)$$

$$e^{\frac{x}{2}} = 1 + \frac{x}{2} + \frac{1}{2}\left(\frac{x}{2}\right)^2 + o(x^2) = 1 + \frac{x}{2} + \frac{1}{8}x^2 + o(x^2)$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + o(x^4)$$

$$\cos\sqrt{x} = 1 - \frac{1}{2}x + \frac{1}{24}x^2 + o(x^2)$$

$$e^{\frac{x}{2}} + \cos\sqrt{x} - 2 = \cancel{1} + \cancel{\frac{x}{2}} + \frac{x^2}{8} + o(x^2) + \cancel{1} - \cancel{\frac{1}{2}x} + \frac{1}{24}x^2 + o(x^2) - \cancel{2}$$
$$= \frac{x^2}{8} + \frac{x^2}{24} + o(x^2) = \frac{1}{6}x^2 + o(x^2)$$

$$\sqrt[3]{1+x} = 1 + \frac{1}{3}x + o(x)$$

$$\sqrt[3]{1+x^2} = 1 + \frac{1}{3}x^2 + o(x^2)$$

$$e^x = 1 + x + o(x)$$

$$e^{x^2} = 1 + x^2 + o(x^2)$$

$$\sqrt[3]{1+x^2} - e^{x^2} = \cancel{1} + \frac{1}{3}x^2 + o(x^2) - \cancel{1} - x^2 - o(x^2)$$
$$= -\frac{2}{3}x^2 + o(x^2)$$

$$\lim_{x\to 0} \frac{e^{\frac{x}{2}} + \cos(\sqrt{x}) - 2}{\sqrt[3]{1+x^2} - e^{x^2}} = \lim_{x\to 0} \frac{\frac{1}{6}x^2 + o(x^2)}{-\frac{2}{3}x^2 + o(x^2)} = \lim_{x\to 0} \frac{\frac{1}{6}x^2}{-\frac{2}{3}x^2} = \frac{\frac{1}{6}}{-\frac{2}{3}} = -\frac{1}{4}$$

4) $$\lim_{x\to 0} \frac{\log(1+x^2) - \frac{x}{2}\sin(2x)}{(e^x - 1 - x)^2}$$

Numeratore:

$$\log(1+x) = x - \frac{x^2}{2} + o(x^2)$$

$$\log(1+x^2) = x^2 - \frac{1}{2}x^4 + o(x^4)$$

$$\sin x = x - \frac{x^3}{6} + o(x^3)$$

$$\sin(2x) = 2x - \frac{(2x)^3}{6} + o(x^3) = 2x - \frac{4}{3}x^3 + o(x^3)$$

$$\frac{x}{2}\sin(2x) = \frac{x}{2}\left(2x - \frac{4}{3}x^3 + o(x^3)\right) = x^2 - \frac{2}{3}x^4 + o(x^4)$$

$$\log(1+x^2) - \frac{x}{2}\sin(2x) = \cancel{x^2} - \frac{1}{2}x^4 + o(x^4) - \cancel{x^2} + \frac{2}{3}x^4 + o(x^4)$$

$$= \left(\frac{2}{3} - \frac{1}{2}\right)x^4 + o(x^4)$$

$$= \frac{1}{6}x^4 + o(x^4)$$

Denominatore:

$$(e^x - 1 - x)^2 = \left(1 + x + \frac{1}{2}x^2 + o(x^2) - 1 - x\right)^2 = \left(\frac{1}{2}x^2 + o(x^2)\right)^2$$

$$= \frac{1}{4}x^4 + o(x^4)$$

$$\lim_{x\to 0} \frac{\log(1+x^2) - \frac{x}{2}\sin(2x)}{(e^x - 1 - x)^2} = \lim_{x\to 0} \frac{\frac{1}{6}x^4 + o(x^4)}{\frac{1}{4}x^4 + o(x^4)} = \frac{2}{3}.$$